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The Waves on a Rotating Liquid Spheroid of Finite Ellipticity

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VI. *The Waves on a Rotating Liquid Spheroid of Finite Ellipticity.*

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1. The hydrodynamical problem of finding the waves or oscillations on a gravitating mass of liquid which, when undisturbed, is rotating as if rigid with finite angular velocity, in the form of an ellipsoid or spheroid, was first successfully attacked by M. Poincaré in 1885. In his important memoir, "Sur l'Équilibre d'une Masse Fluide animée d'un Mouvement de Rotation,"* Poincaré has (§ 13) obtained the differential equations for the oscillations of rotating liquid, and shown that, by a transformation of projection, the determination of the oscillations of any particular period is reducible to finding a suitable solution of Laplace's equation. He then applies Lamé's functions to the case of the ellipsoid, showing that the differential equations are satisfied by a series of Lamé's functions referred to a certain auxiliary ellipsoid, the boundary-conditions, however, involving ellipsoidal harmonics, referred to both the auxiliary and actual fluid ellipsoid. At the same time, Poincaré's analysis does not appear to admit of any definite conclusions being formed as to the nature and frequencies of the various periodic free waves.

The present paper contains an application of Poincaré's methods to the simpler case when the fluid ellipsoid is one of revolution (Maclaurin's spheroid). The solution is effected by the use of the ordinary tesseral or zonal harmonics applicable to the fluid spheroid and to the auxiliary spheroid required in solving the differential equation. The problem is thus freed from the difficulties attending the use of Lamé's functions, and is further simplified by the fact that each independent solution contains harmonics of only one particular degree and rank.

By substituting in the conditions to be satisfied at the surface of the spheroid we arrive at a single boundary-equation. If we are treating the forced tides due to a known periodic disturbing force, this equation determines their amplitude and, hence, the elevation of the tide above the mean surface of the spheroid at any point at any time. If there be no disturbing force, it determines the frequencies of the various free waves determined by harmonics of given order and rank. Denoting by κ the ratio of the frequency of the free waves to twice the frequency of rotation of the

* 'Acta Mathematica,' vol. 7.

liquid about its axis, the values of κ are the roots of a rational algebraic equation, and depend only on the eccentricity of the spheroid, as well as the degree and rank of the harmonic, while the number of different free waves depends on the degree of the equation in κ. At any instant the height of the disturbance at any point of the surface is proportional to the corresponding surface harmonic on the spheroid, multiplied by the central perpendicular on the tangent plane, and is of the same form for all waves determined by harmonics of any given degree and rank, whatever be their frequency; but the motions of the fluid particles in the interior will differ in nature in every case.

Taking first the case of zonal harmonics of the n^{th} degree, we find that, according as n is even or odd, there will be $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ different periodic motions of the liquid. These are essentially oscillatory in character and symmetrical about the axis of the spheroid. In all but one of these the value of κ is essentially less than unity, that is, the period is greater than the time of a semi-revolution of the liquid.

Taking next the tesseral harmonics of degree n and rank s, we find that they determine $n-s+2$ periodic small motions. These are essentially tidal waves rotating with various angular velocities about the axis of the spheroid, the angular velocities of those rotating in opposite directions being in general different. All but two of the values of κ are numerically less than unity, the periods of the corresponding tides at a point fixed relatively to the liquid being greater than the time of a semi-revolution of the mass. The mean angular velocity of these $n-s+2$ waves is less than that of rotation of the mass by $2/\{s(n-s+2)\}$ of the latter.

In the two waves determined by any sectorial harmonic, the relative motion of the liquid particles is irrotational. The harmonics of degree 2 and rank 1 give rise to a kind of precession, of which there are two.

I have calculated the relative frequencies of several of the principal waves on a spheroid whose eccentricity is $\frac{1}{2}\sqrt{2}$.

The question of stability is next dealt with, it being shown that in the present problem, in which the liquid forming the spheroid is supposed perfect, the criteria are entirely different from the conditions of secular stability obtained by POINCARÉ for the case when the liquid possesses any amount of viscosity, and which latter depend on the energy being a minimum. In fact, for a disturbance initially determined by any harmonic (provided that it is symmetrical with respect to the equatorial plane, since for unsymmetrical displacements the spheroid cannot be unstable), the limits of eccentricity consistent with stability are wider for a perfect liquid spheroid than for one possessing any viscosity. If we assume that the disturbed surface initially becomes ellipsoidal, the conditions of stability found by the methods of this paper agree with those of RIEMANN.

The case when the ellipticity and, therefore, the angular velocity are very small is next discussed, it being shown that all but two of the waves, or all but one of the oscillations for any particular harmonic, become unimportant, their periods increasing indefinitely. In the case of those whose periods remain finite for a non-rotating spherical mass, the effect of a small angular velocity ω of the liquid is to cause them to turn round the axis with a velocity less than that of the liquid by ω/n.

Finally, the methods of treating forced tides are further discussed. The general cases of a "semi-diurnal" forced tide, or of permanent deformations due to constant disturbing forces, are mentioned in connection with some peculiarities they present; and these are followed by examples of the determination of the forced tides due to the presence of an attracting mass, first, when the latter moves in any orbit about the spheroid, secondly, when it rotates uniformly about the spheroid in its equatorial plane. The effects of such a body in destroying the equilibrium of the spheroid where the forced tide coincides with one of the free tides form the conclusion of this paper.

POINCARÉ'S *Differential Equations for Waves or Oscillations of Rotating Liquid.*

2. Suppose a mass of gravitating liquid is in relative equilibrium when rotating as if rigid about a fixed axis with angular velocity ω, and that it is required to determine the waves or small oscillations due to a slight disturbance of the mass.

Let the motion be referred to a set of orthogonal moving axes, of which the axis of z is the fixed axis of rotation, while the axes of x, y rotate about it with angular velocity ω. In the steady or undisturbed motion the positions of the fluid particles relative to these axes will remain fixed. In the oscillations, let U, V, W be the small component velocities of the fluid at the point (x, y, z) *relative* to the axes. The actual component velocities referred to axes fixed in space and coinciding with our axes of x, y, z, at the time considered, will be $\mathrm{U} - \omega y$, $\mathrm{V} + \omega x$, W, and the equations of hydrodynamics may be written *

$$\frac{\partial \mathrm{U}}{\partial t} - \omega(\mathrm{V} + \omega x) + \mathrm{U}\frac{\partial \mathrm{U}}{\partial x} + \mathrm{V}\left(\frac{\partial \mathrm{U}}{\partial y} - \omega\right) + \mathrm{W}\frac{\partial \mathrm{U}}{\partial z} = \frac{\partial}{\partial x}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),$$

$$\frac{\partial \mathrm{V}}{\partial t} + \omega(\mathrm{U} - \omega y) + \mathrm{U}\left(\frac{\partial \mathrm{V}}{\partial x} + \omega\right) + \mathrm{V}\frac{\partial \mathrm{V}}{\partial y} + \mathrm{W}\frac{\partial \mathrm{V}}{\partial z} = \frac{\partial}{\partial y}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),$$

$$\frac{\partial \mathrm{W}}{\partial t} + \mathrm{U}\frac{\partial \mathrm{W}}{\partial x} + \mathrm{V}\frac{\partial \mathrm{W}}{\partial y} + \mathrm{W}\frac{\partial \mathrm{W}}{\partial z} = \frac{\partial}{\partial z}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),$$

V_1 being the potential due to the attraction of the liquid and any forces which may act on it, p the pressure, and ρ the density.

For small disturbances we may neglect squares and products of the relative velocities U, V, W (as is usual in wave problems), and, therefore, the above equations reduce to

* BASSET, 'Hydrodynamics,' vol. 1, § 23; or GREENHILL, 'Encyclopædia Britannica,' article "Hydromechanics."

$$\left.\begin{aligned}\frac{\partial U}{\partial t} - 2\omega V &= \frac{\partial \psi}{\partial x},\\ \frac{\partial V}{\partial t} + 2\omega U &= \frac{\partial \psi}{\partial y},\\ \frac{\partial W}{\partial t} \qquad &= \frac{\partial \psi}{\partial z},\end{aligned}\right\} \quad \ldots \ldots \quad (1),$$

where

$$\psi = V_1 - \frac{p}{\rho} + \tfrac{1}{2}\omega^2(x^2 + y^2) \quad \ldots \ldots \quad (2).$$

We have also the equation of continuity,

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \quad \ldots \ldots \quad (3).$$

Eliminating U, V, W from equations (1), (3), we obtain the differential equation

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + 4\omega^2\frac{\partial^2 \psi}{\partial z^2} = 0 \quad \ldots \ldots \quad (4),$$

where, as usual, ∇^2 stands for LAPLACE's operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

3. Let us now consider separately the simple harmonic oscillations of one particular period. Assume that U, V, W, and ψ all vary as $e^{2\iota\omega\kappa t}$, so that the ratio of the period of oscillation to the time of a complete revolution of the liquid mass about its axis is $1/2\kappa$. The equations (1), (4) reduce to

$$\left.\begin{aligned}2\omega(\iota\kappa U - V) &= \frac{\partial \psi}{\partial x},\\ 2\omega(\iota\kappa V + U) &= \frac{\partial \psi}{\partial y},\\ 2\omega\iota\kappa W \qquad &= \frac{\partial \psi}{\partial z},\end{aligned}\right\} \quad \ldots \ldots \quad (5),$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \left(1 - \frac{1}{\kappa^2}\right)\frac{\partial^2 \psi}{\partial z^2} = 0 \quad \ldots \ldots \quad (6).$$

Put

$$1 - \frac{1}{\kappa^2} = \tau^2 \quad \ldots \ldots \quad (7),$$

and

$$z = \tau z' \quad \ldots \ldots \quad (8).$$

Equation (6) now becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z'^2} = 0 \quad \ldots \ldots \quad (9).$$

If κ be greater than unity, τ and, therefore, also z' will be real. We may take (x, y, z') to be the coordinates of a point corresponding to the point (x, y, z) of the liquid. We thus obtain a new region of points derivable from the original region by homogeneous strain parallel to z or by projection. This region may be called the auxiliary region, and the surface formed by points corresponding to points on the fluid surface, the auxiliary surface. Our problem thus reduces to that of finding a suitable value of ψ satisfying LAPLACE's equation (9) within the space bounded by the auxiliary surface.

But we must revert to the original system in order to satisfy the boundary-conditions, which must hold at the actual surface of the liquid, not at the auxiliary surface. If the surface of the liquid be free, p must be constant over it, and, therefore, the condition to be satisfied all over the *disturbed* surface of the liquid is

$$\psi = V_1 + \tfrac{1}{2}\omega^2(x^2 + y^2) + \text{const.} \qquad (10).$$

In forming the expression for V_1 we must remember that the gravitation potential is due to the disturbed configuration of the liquid mass.

If κ be less than unity, τ will be imaginary, and, therefore, the auxiliary surface will also be imaginary. But the results arrived at by this method in the case where τ is real will still hold good even if τ be imaginary, provided that the expression obtained for ψ is a real function of the coordinates x, y, z. The method breaks down if $\kappa = \pm 1$, when τ vanishes; this must be treated as a limiting case.

Solution for the Spheroid by Spheroidal Harmonics.

4. Let the liquid be in the form of a MACLAURIN's spheroid the equation of whose surface is

$$\frac{x^2 + y^2}{c^2(\zeta_0^2 + 1)} + \frac{z^2}{c^2\zeta_0^2} \equiv \frac{x^2 + y^2}{c^2 \operatorname{cosec}^2 \alpha} + \frac{z^2}{c^2 \cot^2 \alpha} = 1 \qquad (11),$$

so that

$$\zeta_0 = \cot \alpha \qquad (12),$$

and $\sin \alpha$ is the eccentricity of the spheroid, c being the radius of its focal circle.

The locus of the corresponding point (x, y, z') is the auxiliary quadric

$$\frac{x^2 + y^2}{c^2 \operatorname{cosec}^2 \alpha} + \frac{\tau^2 z'^2}{c^2 \cot^2 \alpha} = 1 \qquad (13).$$

This quadric will be a prolate spheroid if τ^2 lies between zero and $\cos^2 \alpha$, that is, if κ^2 lies between unity and $\operatorname{cosec}^2 \alpha$. If τ^2 is greater than $\cos^2 \alpha$, or κ^2 greater than $\operatorname{cosec}^2 \alpha$, the spheroid will be oblate. If τ^2 be negative, or κ^2 less than unity, equation (13) represents a hyperboloid of one sheet, but the part corresponding to the liquid surface is the imaginary portion for which $x^2 + y^2$ is less than $c^2 \operatorname{cosec}^2 \alpha$, and z'^2 is negative; this is the imaginary auxiliary spheroid.

We shall take as our standard case that in which equation (13) represents a prolate auxiliary spheroid. Let it be written in the form

$$\frac{x^2 + y^2}{k^2 (\nu_0^2 - 1)} + \frac{z'^2}{k^2 \nu_0^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

so that

$$k^2 (\nu_0^2 - 1) = c^2 \operatorname{cosec}^2 \alpha,$$

$$k^2 \nu_0^2 = \frac{c^2 \cot^2 \alpha}{\tau^2} = \frac{c^2 \cot^2 \alpha \,.\, \kappa^2}{\kappa^2 - 1}.$$

Solving for ν_0, k, we find

$$\nu_0 = \frac{\kappa \cos \alpha}{\surd (1 - \kappa^2 \sin^2 \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$k^2 = c^2 \frac{\operatorname{cosec}^2 \alpha - \kappa^2}{\kappa^2 - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

The solution of the differential equation (9) must be effected by means of spheroidal harmonics applicable to the auxiliary spheroid (14), whilst the expressions for the gravitation potential of the liquid mass and the boundary-conditions will involve spheroidal harmonics referred to the actual liquid spheroid (11). We must, therefore, use two different sets of orthogonal elliptic coordinates for the auxiliary and the actual systems. Let these coordinates be denoted by (μ', ν, ϕ) (μ, ζ, ϕ) respectively, and let them be connected with the rectangular coordinates in the two systems by the relations

$$\left.\begin{aligned} x &= k \surd (\nu^2 - 1) \surd (1 - \mu'^2) \cos \phi = c \surd (\zeta^2 + 1) \surd (1 - \mu^2) \cos \phi \\ y &= k \surd (\nu^2 - 1) \surd (1 - \mu'^2) \sin \phi = c \surd (\zeta^2 + 1) \surd (1 - \mu^2) \sin \phi \\ z' &= k\nu\mu' \qquad\qquad\qquad\qquad\qquad = c\zeta\mu/\tau \\ &\text{and, therefore,} \\ z &= k\nu\mu'\tau \qquad\qquad\qquad\qquad\quad\; = c\zeta\mu \end{aligned}\right\} \quad . \quad (17).$$

The surfaces of the spheroids will be given by the equations

$$\zeta = \zeta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

or

$$\nu = \nu_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\text{A});$$

moreover, all over these surfaces, at corresponding points,

$$\mu = \mu' \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

The angular coordinate ϕ is the same in both systems; but, except over the surfaces, μ will not be equal to μ', nor will any other two of the surfaces $\nu =$ constant and $\zeta =$ constant coincide.

Put $\mu' = \cos\theta$. On transforming to (θ, ν, ϕ), equation (9) becomes

$$\frac{\partial}{\partial \nu}\left\{(\nu^2 - 1)\frac{\partial \psi}{\partial \nu}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{\nu^2 - \cos^2\theta}{(\nu^2 - 1)\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0 \quad . \quad . \quad (20),$$

of which a solution, finite and continuous at all points within the spheroid (14), is

$$\psi = A_n^s T_n^{(s)}(\mu') \, T_n^s(\nu) \, e^{\iota s \phi} e^{2\iota\omega\kappa t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

where A_n^s is any constant, and

$$T_n^{(s)}(\mu') = (1 - \mu'^2)^{s/2}\left(\frac{d}{d\mu'}\right)^s P_n(\mu') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$T_n^s(\nu) = (\nu^2 - 1)^{s/2}\left(\frac{d}{d\nu}\right)^s P_n(\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

P_n denoting the zonal harmonic of degree n.

In our standard case ν is real and greater than unity, and in every case μ' lies between the limits $+1$ and -1, and is real. I have adopted the above notation (according to which the functions $T_n^{(s)}$, T_n^s differ in form by the constant factor $(-1)^{s/2}$) in order to avoid introducing imaginary coefficients unnecessarily.*

It is easy to see that the solution (21) is applicable in every case. For, if τ^2 be greater than $\cos^2\alpha$, both k and ν are purely imaginary; whilst, if τ^2 be negative, we may show that k will be imaginary, but ν will be real and less than unity. In any case $T_n^s(\nu)$ will be either real or purely imaginary, so that $A_n^s T_n^{(s)}(\mu) \, T_n^s(\nu)$ can be always made a *real* function of the coordinates (x, y, z). Moreover, the right-hand side of (21) is finite, single valued, and continuous throughout the liquid spheroid, and satisfies the differential equation (6). It therefore only remains to investigate the boundary-conditions which must be satisfied by ψ at the surface of the liquid.

5. The spheroidal harmonics referred to the liquid spheroid, required for these boundary-conditions, will be formed as follows :—

Let

$$p_n(\zeta) = \frac{1}{2^n n!}\left(\frac{d}{d\zeta}\right)^n (\zeta^2 + 1)^n \equiv (-1)^{n/2} P_n(\iota\zeta) \quad . \quad . \quad . \quad (24),$$

$$t_n^s(\zeta) = (\zeta^2 + 1)^{s/2}\left(\frac{d}{d\zeta}\right)^s p_n(\zeta) \equiv (-1)^{n/2} T_n^s(\iota\zeta) \quad . \quad . \quad . \quad (25),$$

* The tesseral harmonics may be replaced by the associated functions of the first kind of HEINE without any change in the formulæ, the constant coefficients being supposed included in A_n^s.

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and let

$$q_n(\zeta) = p_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{p_n(\zeta)\}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

$$u_n^s(\zeta) = t_n^s(\zeta)\int_\zeta^\infty \frac{d\zeta}{(\zeta^2+1)\{t_n^s(\zeta)\}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Then the expressions

$$v_1 = \mathrm{B}_n{}^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n{}^{(s)}(\mu)\, t_n{}^s(\zeta)/t_n{}^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$v_0 = \mathrm{B}_n{}^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n{}^{(s)}(\mu)\, u_n{}^s(\zeta)/u_n{}^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

are solutions of LAPLACE's equation which are finite and continuous, the former throughout the interior of the spheroid ($\zeta = \zeta_0$), the latter throughout all space outside the spheroid, and vanishing at infinity; while at the surface ($\zeta = \zeta_0$) both expressions become equal to

$$[v] = \mathrm{B}_n{}^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n{}^{(s)}(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

The Elevation of the Waves on the Surface.

6. Let h be the normal displacement at any point of the liquid surface, *i.e.*, the height of the wave above the level of the undisturbed spheroid.

Let ϖ be the central perpendicular on the tangent plane, and $d\mathrm{N}$ an element of the outward drawn normal to the surface of the *liquid* spheroid (11). We readily find

$$\frac{\partial x}{\partial\zeta} = \frac{\zeta x}{\zeta^2+1}, \qquad \frac{\partial y}{\partial\zeta} = \frac{\zeta y}{\zeta^2+1}, \qquad \frac{\partial z}{\partial\zeta} = \frac{\zeta z}{\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

and at the surface, since the element $d\mathrm{N}$ is a tangent to the curve $\mu =$ const., $\phi =$ const.; therefore,

$$\left(\frac{d\mathrm{N}}{d\zeta}\right)^2 = \left(\frac{\partial x}{\partial\zeta}\right)^2 + \left(\frac{\partial y}{\partial\zeta}\right)^2 + \left(\frac{\partial z}{\partial\zeta}\right)^2;$$

whence,

$$\frac{d\mathrm{N}}{d\zeta} = c\sqrt{\left(\frac{\zeta_0{}^2+\mu^2}{\zeta_0{}^2+1}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

the differentials in (32) being *total*

Also

$$\varpi = c\zeta_0\sqrt{\left(\frac{\zeta_0{}^2+1}{\zeta_0{}^2+\mu^2}\right)},$$

so that

$$\varpi\, d\mathrm{N} = c^2\zeta_0\, d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

From equations (5) we obtain

$$\left.\begin{aligned} 2\omega(1-\kappa^2)\,\mathrm{U} &= \iota\kappa\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y} \\ 2\omega(1-\kappa^2)\,\mathrm{V} &= \iota\kappa\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x} \\ 2\omega(1-\kappa^2)\,\mathrm{W} &= \iota\kappa\tau^2\frac{\partial\psi}{\partial z} \end{aligned}\right\} \quad \ldots \quad (34).$$

Multiplying by $\partial x/\partial \mathrm{N}$, $\partial y/\partial \mathrm{N}$, $\partial z/\partial \mathrm{N}$, and adding, we find at the surface

$$2\omega(1-\kappa^2)\left(\mathrm{U}\frac{\partial x}{\partial \mathrm{N}}+\mathrm{V}\frac{\partial y}{\partial \mathrm{N}}+\mathrm{W}\frac{\partial z}{\partial \mathrm{N}}\right)$$
$$=\iota\kappa\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial \mathrm{N}}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial \mathrm{N}}+\tau^2\frac{\partial\psi}{\partial z}\frac{\partial x}{\partial \mathrm{N}}\right)+\left(\frac{\partial\psi}{\partial y}\frac{\partial x}{\partial \mathrm{N}}-\frac{\partial\psi}{\partial x}\frac{\partial y}{\partial \mathrm{N}}\right).$$

But $\mathrm{U}\partial x/\partial \mathrm{N}+\mathrm{V}\partial y/\partial \mathrm{N}+\mathrm{W}\partial z/\partial \mathrm{N}$ is the normal velocity of the liquid relative to the moving axes, and is therefore equal to $\partial h/\partial t$ or to $2\iota\omega\kappa h$. We have, therefore,

$$4\omega^2(1-\kappa^2)\kappa h$$
$$=\kappa\frac{d\zeta}{d\mathrm{N}}\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\zeta}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\zeta}+\tau^2\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial\zeta}\right)-\iota\frac{\partial\zeta}{\partial \mathrm{N}}\left(\frac{\partial\psi}{\partial y}\frac{\partial x}{\partial\zeta}-\frac{\partial\psi}{\partial x}\frac{\partial y}{\partial\zeta}\right)$$
$$=\kappa c^2\zeta_0\frac{d\zeta}{d\mathrm{N}}\left(\frac{x}{c^2(\zeta_0^2+1)}\frac{\partial\psi}{\partial x}+\frac{y}{c^2(\zeta_0^2+1)}\frac{\partial\psi}{\partial y}+\frac{\tau^2 z}{c^2\zeta_0^2}\frac{\partial\psi}{\partial z}\right)-\frac{\iota\zeta_0}{\zeta_0^2+1}\frac{d\zeta}{d\mathrm{N}}\left(x\frac{\partial\psi}{\partial y}-y\frac{\partial\psi}{\partial x}\right)$$
$$=\kappa c^2\zeta_0\frac{d\zeta}{d\mathrm{N}}\left(\frac{x}{k^2(\nu_0^2-1)}\frac{\partial\psi}{\partial x}+\frac{y}{k^2(\nu_0^2-1)}\frac{\partial\psi}{\partial y}+\frac{z'}{k^2\nu_0^2}\frac{\partial\psi}{\partial z'}\right)-\frac{\iota\zeta_0}{\zeta_0^2+1}\frac{d\zeta}{d\mathrm{N}}\frac{\partial\psi}{\partial\phi}$$
$$=\zeta_0\frac{d\zeta}{d\mathrm{N}}\left\{\frac{\kappa c^2}{\nu_0 k^2}\frac{\partial\psi}{\partial\nu}-\frac{\iota}{\zeta_0^2+1}\frac{\partial\psi}{\partial\phi}\right\} \quad \ldots \quad (35).$$

Now, taking ψ as given by (21), we have

$$\frac{d\psi}{d\phi}=\iota s\psi \quad \ldots \quad (36);$$

moreover, since by (23)

$$\mathrm{T}_n^s(\nu)=(\nu^2-1)^{s/2}\,\mathrm{D}^s\mathrm{P}_n(\nu),$$

where the symbol D stands for differentiation with respect to ν, therefore

$$\mathrm{DT}_n^s(\nu)=(\nu^2-1)^{s/2}\{\mathrm{D}^{s+1}\mathrm{P}_n(\nu)+s\nu/(\nu^2-1)\,.\,\mathrm{D}^s\mathrm{P}_n(\nu)\} \quad \ldots \quad (37).$$

Also at the surface μ' is equal to μ.

Hence, we find

$$4\omega^2(1-\kappa^2)\kappa h = A_n^s\zeta_0\frac{d\zeta}{dN}\left\{\frac{\kappa c^2}{\nu_0^2k^2}\nu_0 D^{s+1}P_n(\nu_0) + \frac{s\kappa c^2}{k^2(\nu_0^2-1)}D^sP_n(\nu_0) + \frac{s}{\zeta_0^2+1}D^sP_n(\nu_0)\right\}$$
$$\times(\nu_0^2-1)^{s/2}T_n^{(s)}(\mu)e^{\iota s\phi}e^{2\iota\omega\kappa t}$$
$$= A_n^s\zeta_0\frac{d\zeta}{dN}\left\{\frac{\kappa^2-1}{\kappa\zeta_0^2}\nu_0 D^{s+1}P_n(\nu_0) + \frac{s(\kappa+1)}{\zeta_0^2+1}D^sP_n(\nu_0)\right\}$$
$$\times(\nu_0^2-1)^{s/2}T_n^{(s)}(\mu)e^{\iota s\phi}e^{2\iota\omega\kappa t}$$

by (15), (16).

Whence

$$h = C_n^s\varpi T_n^{(s)}(\mu)e^{\iota s\phi}e^{2\iota\omega\kappa t} \quad \ldots \quad (38),$$

where

$$C_n^s = -A_n^s\frac{\tan^2\alpha}{4\omega^2\kappa c^2}\left\{\frac{1}{\kappa}\nu_0 D^{s+1}P_n(\nu_0) + \frac{s\cos^2\alpha}{\kappa-1}D^sP_n(\nu_0)\right\}(\nu_0^2-1)^{s/2} \quad . \quad (39);$$

moreover, the equation of the disturbed surface of the liquid is

$$\zeta = \zeta_0 + \delta\zeta_0 \quad \ldots \quad (40),$$

where

$$\delta\zeta_0 = h\frac{d\zeta}{dN} = C_n^s\varpi^2\tan\alpha/c^2\,T_n^{(s)}(\mu)e^{\iota s\phi}e^{2\iota\omega\kappa t}. \quad \ldots \quad (41).$$

The Boundary-Conditions.

7. Let V_0 be the potential of a mass of the liquid filling the spheroid

$$\zeta = \zeta_0 \quad \ldots \quad (18),$$

and let v be the potential of a distribution of the liquid of thickness everywhere equal to h over the surface of the spheroid. The combination of these two distributions is equivalent to the liquid mass as disturbed by the waves, so that

$$V_1 = V_0 + v \quad \ldots \quad (42).$$

For the free waves, the boundary-equation (10) requires that

$$\psi = V_0 + v + \tfrac{1}{2}\omega^2(x^2+y^2) + \text{const.} \quad \ldots \quad (43),$$

all over the surface (40).

Now ψ, v, $\delta\zeta_0$ are small quantities of the first order. Hence, expanding by TAYLOR's theorem, we have to first order

$$[\psi] = [V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + [v] + \delta\zeta_0 \frac{d}{d\zeta}\{V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} + \text{const.} \quad (44),$$

where the square brackets indicate that ζ is to be put equal to ζ_0.

In the case of forced tides due partly to small disturbing forces whose potential at any instant is V_2, and partly to periodic variations of pressure p_2 over the surface of the liquid, the condition at the surface becomes

$$[\psi] = [V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + [v] + \delta\zeta_0 \frac{d}{d\zeta}\{V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} + \left[V_2 - \frac{p_2}{\rho}\right] + \text{const.} \quad . \quad (44^*)$$

Equating to zero the non-periodic terms, we obtain the well-known condition for steady motion

$$[V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + \text{const.} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Here

$$V_0 = \text{const.} - \tfrac{1}{2}\{A(x^2+y^2) + Cz^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

where

$$\left.\begin{aligned} A &= 4\pi\rho\gamma\,\zeta_0(\zeta_0^2+1)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2+1)^2} = 4\pi\rho\gamma\,\text{cosec}^2\alpha\cot\alpha\,\frac{u_1^1(\zeta_0)}{t_1^1(\zeta_0)} \\ C &= 4\pi\rho\gamma\,\zeta_0(\zeta_0^2+1)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2+1)\,\zeta^2} = 4\pi\rho\gamma\,\text{cosec}^2\alpha\cot\alpha\,\frac{q_1(\zeta_0)}{p_1(\zeta_0)} \end{aligned}\right\} \quad . \quad (47),$$

γ being the constant of gravitation, and being put equal to unity if the density is expressed in astronomical units.

From (45) we have, in the usual manner,

$$(A-\omega^2)(x^2+y^2) + Cz^2 = C\zeta_0^2\{(x^2+y^2)/(\zeta_0^2+1) + z^2/\zeta_0^2\} \quad . \quad . \quad (48),$$

whence

$$\begin{aligned} \omega^2 &= A - C\zeta_0^2/(\zeta_0^2+1) \\ &= 4\pi\rho\gamma\,\zeta_0\{t_1^1(\zeta_0)\,u_1^1(\zeta_0) - p_1(\zeta_0)\,q_1(\zeta_0)\} \quad . \quad . \quad . \quad . \quad . \quad (49), \end{aligned}$$

which can also be put in the form

$$\begin{aligned} \omega^2 &= 4\pi\rho\gamma\,\zeta_0\{\tfrac{1}{2}(3\zeta_0^2+1)\cot^{-1}\zeta_0 - \tfrac{3}{2}\zeta_0\} \\ &= 4\pi\rho\gamma\,\zeta_0 q_2(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50). \end{aligned}$$

From (44*), (45) the boundary-condition for the oscillations is

$$[\psi] = [v] + \delta\zeta_0\frac{\partial}{\partial\zeta}\{V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} + \left[V_2 - \frac{p_2}{\rho}\right] \quad . \quad . \quad . \quad (51).$$

Now, by (48)

$$\delta\zeta_0 \frac{d}{d\zeta}\{V_0 + \tfrac{1}{2}\omega^2(x^2 + y^2)\}$$
$$= -\tfrac{1}{2}C\zeta_0^2\,\delta\zeta_0\left\{\frac{\partial x}{\partial\zeta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\zeta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\zeta}\frac{\partial}{\partial z}\right\}\left(\frac{x^2+y^2}{\zeta_0^2+1} + \frac{z^2}{\zeta_0^2}\right)$$
$$= -4\pi\rho\gamma(\zeta_0^2+1)\,\zeta_0^4\frac{q_1(\zeta_0)}{p_1(\zeta_0)}\left\{\frac{x^2+y^2}{(\zeta_0^2+1)^2} + \frac{z^2}{\zeta_0^4}\right\}\delta\zeta_0$$
$$= -4\pi\rho\gamma(\zeta_0^2+1)\,\zeta_0^2 p_1(\zeta_0)\,q_1(\zeta_0)\,c^4\,\delta\zeta_0/\varpi^2$$
$$= -4\pi\rho\gamma\,C_n^s c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,.\,p_1(\zeta_0)\,q_1(\zeta_0)\,T_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t} \quad . \quad . \quad (52).$$

8. To find v.—Suppose that the values of this potential inside and outside the spheroid respectively are given by the formulæ

$$v_1 = B_n^s e^{2\iota\omega\kappa t}e^{\iota s\phi}T_n^{(s)}(\mu)\,t_n^s(\zeta)/t_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$v_0 = B_n^s e^{2\iota\omega\kappa t}e^{\iota s\phi}T_n^{(s)}(\mu)\,u_n^s(\zeta)/u_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Since v_1 v_0 are due to a surface distribution of surface density ρh, therefore,

$$-4\pi\rho\gamma h = \left[\frac{\partial V_0}{\partial N}\right] - \left[\frac{\partial V_1}{\partial N}\right] = \frac{d\zeta}{dN}\left[\frac{\partial V_0}{\partial\zeta} - \frac{\partial V_1}{\partial\zeta}\right]_{(\zeta=\zeta_0)}$$
$$= B_n^s\frac{d\zeta}{dN}e^{2\iota\omega\kappa t}e^{\iota s\phi}T_n^{(s)}(\mu)\left\{\frac{Du_n^s(\zeta_0)}{u_n^s(\zeta_0)} - \frac{Dt_n^s(\zeta_0)}{t_n^s(\zeta_0)}\right\}$$
$$= -B_n^s\frac{d\zeta}{dN}e^{2\iota\omega\kappa t}e^{\iota s\phi}T_n^{(s)}(\mu) \div \{(\zeta_0^2+1)\,t_n^s(\zeta_0)\,u_n^s(\zeta_0)\},$$

and, therefore, at the surface, by (30)

$$[v] \equiv B_n^s e^{2\iota\omega\kappa t}e^{\iota s\phi}T_n^{(s)}(\mu)$$
$$= 4\pi\rho\gamma\,C_n^s c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,.\,t_n^s(\zeta_0)\,u_n^s(\zeta_0)\,T_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t} \quad . \quad . \quad . \quad . \quad (53).$$

9. Lastly, in the forced oscillations, whatever be the variable conservative bodily forces or surface tractions producing them, we know that it is always possible to expand the value of $[V_2 - p_2/\rho]$ over the surface of the spheroid and at all times in a series of the form

$$[V_2 - p_2/\rho] = \Sigma_{\kappa=-\infty}^{\kappa=\infty}\Sigma_{n=1}^{n=\infty}\Sigma_{s=0}^{s=n}W^s_{(n,\kappa)}T_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t} \quad . \quad . \quad . \quad . \quad (54),$$

where $W^s_{(n,\kappa)}$ is a constant, and the summations may extend to all possible values of κ, but only to integral values of n and s. The effect of each term may be considered separately. To do this, let us take the case when there is a single term only, *i.e.*, take

$$[V_2 - p_2/\rho] = W^s_{(n,\kappa)}T_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

In the waves produced the values of n, s, κ will be the same.

Substituting from (21), (52), (53), (55) in (51), we obtain

$$A_n^s(\nu_0^2-1)^{s/2}\,D^sP_n(\nu_0)+4\pi\rho\gamma\,C_n^s c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,\{p_1(\zeta_0)\,q_1(\zeta_0)-t_n^s(\zeta_0)\,u_n^s(\zeta_0)\} = W^s_{(n,\kappa)} \quad . \quad . \quad (56).$$

This equation, combined with (39), suffices to determine the unknown constants A_n^s, C_n^s in terms of the known coefficient $W^s_{(n,\kappa)}$, and thus the amplitude of the forced oscillation is determined in terms of that of the disturbing force.

10. The most interesting point is to determine C_n^s, in order to find the height of the corrugations on the surface. This plan has, moreover, the advantage that, in considering the effect of several disturbing forces of different periods, we may add together the elevations (h) due to the separate forces, whereas, in determining the value of ψ, the terms having different periods are referred to different auxiliary systems. Substituting for A_n^s in terms of C_n^s from (39) in (56), and writing, for brevity,

$$K_n^s(\zeta_0) = p_1(\zeta_0)\,q_1(\zeta_0) - t_n^s(\zeta_0)\,u_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

$$M = \tfrac{4}{3}\pi\rho c^3\cot\alpha\,\mathrm{cosec}^2\alpha = \text{mass of spheroid} \quad . \quad . \quad . \quad (58),$$

we find, after several reductions, the required equation for C_n^s, viz.,

$$\frac{3M\gamma C_n^s}{c}\left\{K_n^s(\zeta_0) - \frac{4\kappa q_2(\zeta_0)\,D^sP_n(\nu_0)}{s\,D^sP_n(\nu_0)/(\kappa-1)+\sec^2\alpha\,.\,\nu_0\,D^{s+1}P_n(\nu_0)/\kappa}\right\} = W^s_{(n,\kappa)} \quad . \quad (59),$$

in which it must be remembered that

$$\nu_0 = \frac{\kappa\cos\alpha}{\sqrt{(1-\kappa^2\sin^2\alpha)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

The Period-Equations for Free Waves.

11. If the oscillations of the liquid be free, we must put $W^s_{(n,\kappa)}$ equal to zero in (59), and we therefore obtain

$$K_n^s(\cot\alpha) - \frac{4\kappa q_2(\cot\alpha)\,D^sP_n(\nu_0)}{s\,D^sP_n(\nu_0)/(\kappa-1)+\sec^2\alpha\,.\,\nu_0\,D^{s+1}P_n(\nu_0)/\kappa} = 0 \quad . \quad . \quad (60),$$

which, together with (15), determines the admissible values of κ and ν_0. In reducing (60) to a rational algebraic equation for κ we must distinguish three cases.

I. Let $s = 0$, and let n be even. Then we know that $P_n(\nu_0)$ and $DP_n(\nu_0)$ contain only even and odd powers of ν_0 respectively, and, therefore, that $DP_n(\nu_0)$ is divisible by ν_0. Multiplying (60) by $DP_n(\nu_0)/\nu_0$, we find (writing K_n for K_n^0)

$$K_n(\cot\alpha)\,DP_n(\nu_0)/\nu_0 - 4q_2(\cot\alpha)\,(1-\kappa^2\sin^2\alpha)\,P_n(\nu_0) = 0 \quad . \quad . \quad (61).$$

Expanding $P_n(\nu_0)$ and $DP_n(\nu_0)/\nu_0$ in powers of ν_0, substituting for ν_0 by means of (15), and multiplying the resulting equation throughout by $(1 - \kappa^2 \sin^2 \alpha)^{n/2-1}$, we find

$$4q_2(\cot\alpha)\left\{(\kappa\cos\alpha)^n - \frac{n(n-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-2}(1-\kappa^2\sin^2\alpha) + \ldots\right\}$$
$$- nK_n(\cot\alpha)\left\{(\kappa\cos\alpha)^{n-2} - \frac{(n-1)(n-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-4}(1-\kappa^2\sin^2\alpha) + \ldots\right\}$$
$$= 0 \quad . \quad . \quad (62).$$

This is a rational algebraic equation in κ of the n^{th} degree, involving only even powers of κ. It is, therefore, satisfied by n values of κ occurring in pairs corresponding to $\frac{1}{2}n$ values of κ^2.

II. Let $s = 0$, but let n be odd. Then $DP_n(\nu_0)$ is not divisible by ν_0. Hence, we must multiply the equation (60) throughout by $DP_n(\nu_0)$ and obtain

$$K_n(\cot\alpha)\,DP_n(\nu_0) - 4q_2(\cot\alpha)(1-\kappa^2\sin^2\alpha)\,\nu_0 P_n(\nu_0) = 0 \quad . \quad . \quad (63).$$

If this be developed in the same manner as in the preceding case, we shall obtain

$$4q_2(\cot\alpha)\left\{(\kappa\cos\alpha)^{n+1} - \frac{n(n-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-1}(1-\kappa^2\sin^2\alpha) + \ldots\right\}$$
$$- nK_n(\cot\alpha)\left\{(\kappa\cos\alpha)^{n-1} - \frac{(n-1)(n-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-3}(1-\kappa^2\sin^2\alpha) + \ldots\right\}$$
$$= 0 \quad . \quad . \quad (64).$$

This is satisfied by $n + 1$ values of κ, but, as before, the positive and negative roots are numerically equal, so that there will only be $\frac{1}{2}(n + 1)$ different values of κ^2.

III. Let s be different from zero. Multiplying by the expression

$$sD^sP_n(\nu_0) + \sec^2\alpha\,.\,(\kappa - 1)/\kappa\,.\,\nu_0 D^{s+1}P_n(\nu_0),$$

we find

$$\sec^2\alpha\, K_n^s(\cot\alpha)(\kappa-1)/\kappa\,.\,\nu_0 D^{s+1}P_n(\nu_0)$$
$$- \{4q_2(\cot\alpha)\kappa(\kappa-1) - sK_n^s(\cot\alpha)\}\,D^sP_n(\nu_0) = 0 \quad . \quad . \quad (65),$$

which reduces to the following equation in κ—

$$\{4q_2(\cot\alpha)\kappa(\kappa-1) - sK_n^s(\cot\alpha)\}\left\{(\kappa\cos\alpha)^{n-s}\right.$$
$$\left. - \frac{(n-s)(n-s-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-s-2}(1-\kappa^2\sin^2\alpha) + \ldots\right\}$$
$$- (n-s)\sec\alpha\,.\,K_n^s(\cot\alpha)(\kappa-1)\left\{(\kappa\cos\alpha)^{n-s-1}\right.$$
$$\left. - \frac{(n-s-1)(n-s-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-s-3}(1-\kappa^2\sin^2\alpha) + \ldots\right\} = 0 \quad . \quad (66).$$

This equation is of the degree $n - s + 2$, and involves both odd and even powers of κ. It therefore has $n - s + 2$ roots, but in the present case these roots do not occur in pairs of equal and opposite values.

Equations (62), (64), (66) are the period-equations of the various free harmonic waves or oscillations of the liquid spheroid. Their roots depend on the value of α or the eccentricity ($\sin \alpha$) alone. The periods of the waves are the corresponding values of $\pi/\omega\kappa$ and depend also on ω.

Nature of the Real Oscillations and Waves.

12. The periodic movements determined by zonal harmonics ($s = 0$) and those determined by tesseral harmonics differ in character considerably.

The former are symmetrical about the axis. Taking the solution

$$\psi = A_n P_n(\mu') P_n(\nu) e^{2\iota\omega\kappa t},$$
$$h = C_n \varpi P_n(\mu) e^{2\iota\omega\kappa t},$$

another solution got by changing the sign of κ is given by

$$\psi = A_n P_n(\mu') P_n(\nu) e^{-2\iota\omega\kappa t},$$
$$h = C_n \varpi P_n(\mu) e^{-2\iota\omega\kappa t}.$$

Compounding these, we get the real motions of the liquid determined by

$$\left.\begin{aligned} \psi &= A_n P_n(\mu') P_n(\nu) \sin(2\omega\kappa t - \epsilon_n) \\ h &= C_n \varpi P_n(\mu) \sin(2\omega\kappa t - \epsilon_n) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67),$$

ϵ_n being any constant.

These are *stationary oscillations* of the liquid about the spheroidal form. By what has already been shown, there are either $\frac{1}{2}n$ or $\frac{1}{2}(n + 1)$ such free oscillations, according as n is even or odd. In all of these oscillations the expression for h is of the same form, that is, the corrugations produced on the surface are similar in each. But this will not be the case with the values of ψ, because the auxiliary systems of spheroidal coordinates to which they are referred are different for each different value of κ. Thus, the motions of the fluid particles in the interior of the mass are different for each of the oscillations.

13. Taking next the case when s is different from zero, let us change the sign of $\sqrt{-1}$ *everywhere* that it occurs in our investigations. The results will still hold good when this is done. Hence, for every root of (66) we get two solutions of the equations of oscillation, giving respectively

$$\psi = A_n^s T_n^{(s)}(\mu') T_n^s(\nu) e^{\iota(s\phi + 2\omega\kappa t)},$$
$$h = C_n^s \varpi T_n^{(s)}(\mu) e^{\iota(s\phi + 2\omega\kappa t)},$$

and also

$$\psi = A_n^s T_n^{(s)}(\mu') T_n^s(\nu) e^{-\iota(s\phi + 2\omega\kappa t)},$$
$$h = C_n^s \varpi T_n^{(s)}(\mu) e^{-\iota(s\phi + 2\omega\kappa t)},$$

which combine to give the real motions

$$\left.\begin{aligned} \psi &= A_n^s T_n^{(s)}(\mu') T_n^s(\nu) \sin(s\phi + 2\omega\kappa t - \epsilon_n^s) \\ h &= C_n^s \varpi T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_n^s) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (68).$$

These represent a system of *waves* travelling round the axis of the spheroid with *relative* angular velocity $-2\omega\kappa/s$. But it must be remembered that the coordinate axes to which we have referred the wave motion are themselves rotating with angular velocity ω. Hence, the angular velocity of the waves in space is $\omega(1 - 2\kappa/s)$. According to our convention, positive values of κ give waves rotating more slowly than the liquid, and *vice versâ*.

There are $n - s + 2$ such waves determined by harmonics of degree n and rank s, and, since the values of κ are not equal and opposite in pairs, these waves do not combine into oscillations fixed relatively to the moving axes. As in the symmetrical oscillations, the form of the corrugations is the same for all the waves, but the motion of the fluid particles different in each.

14. If $\kappa_1, \kappa_2, \ldots, \kappa_{n-s+2}$ be the roots of (66), it is obvious that

$$\kappa_1 + \kappa_2 + \ldots + \kappa_{n-s+2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

Hence, the *mean* relative angular velocity of *all* the different harmonic waves of degree n and rank s is

$$\frac{2\omega}{(n - s + 2)s},$$

in direction opposite to that of rotation of the liquid, whilst their mean actual angular velocity in space is

$$\omega\left\{1 - \frac{2}{(n - s + 2)s}\right\}.$$

Analysis of the Period-Equations.

15. From POINCARÉ's investigations it appears that the spheroid will be secularly stable, even if the liquid be viscous, provided that the coefficients which are here denoted by $K_n^s(\zeta_0)$ or

$$p_1(\zeta) q_1(\zeta) - t_n^s(\zeta) u_n^s(\zeta)$$

are positive for all values of n greater than unity.* From our equations (49), (50) we have

$$-K_1^1(\zeta) = q_2(\zeta) = \omega^2/(4\pi\rho\gamma\,\zeta),$$

so that $K_1^1(\zeta)$ is essentially negative and $q_2(\zeta)$ positive.

In accordance with this, we shall now show that, if $K_n^s(\zeta)$ and $q_2(\zeta)$ be both positive, the roots of the period-equation for harmonic waves of degree n and rank s are all real, and we shall find their situations.

In the first place, let us suppose s is different from zero. The period-equation (66), as it stands, may be written

$$F(\kappa) = 0,$$

where

$$NF(\kappa) \equiv (1-\kappa^2\sin^2\alpha)^{(n-s)/2}\{4q_2(\cot\alpha)\,\kappa(\kappa-1) - sK_n^s(\cot\alpha)\}\,D^sP_n(\nu) - (1-\kappa^2\sin^2\alpha)^{(n-s-1)/2}\sec\alpha\,K_n^s(\cot\alpha)(\kappa-1)\,D^sP_n(\nu),$$

if we write for brevity

$$N \equiv (2n)!/\{2^n n!\,(n-s)!\}.$$

We know that the roots of the equation

$$D^sP_n(\nu) = 0 \quad \ldots \ldots \ldots \ldots \quad (70)$$

are all real, and lie between $+1$ and -1; also they are separated by those of

$$D^{s+1}P_n(\nu) = 0.$$

Let $\kappa_1, \kappa_2, \ldots \kappa_{n-s}$ be the values of κ (taken in descending order of magnitude) corresponding to the roots of (70). These values of κ all lie between $+1$ and -1, also ν decreases as κ decreases. Moreover, if κ is put in turn equal to $\kappa_1, \kappa_2, \ldots \kappa_{n-s}$, the corresponding values of $D^{s+1}P_n(\nu)$ are alternately positive and negative.

We are now in a position to trace the changes in $F(\kappa)$ as κ decreases from $+\infty$ to $-\infty$.

When κ is greater than cosec α, ν is imaginary. But $F(\kappa)$ when written in the form of the left-hand side of (66) is obviously real; also when $\kappa = \infty$ the sign of $F(\kappa)$ is that of the coefficient of κ^{n-s+2}. It is therefore *positive.*

When κ passes through the value cosec α, ν becomes infinite and then becomes real, but $F(\kappa)$ does not in general change sign.

When $\kappa = 1$, $\nu = 1$, $D^sP_n(\nu)$ is positive, and $F(\kappa)$ is negative.

When $\kappa = \kappa_1$, $F(\kappa)$ is positive.

When $\kappa = \kappa_2$, $F(\kappa)$ is negative.

When $\kappa = \kappa_3$, $F(\kappa)$ is positive.

and so on; thus, when $\kappa = \kappa_{n-s}$, $F(\kappa)$ has the same sign as $\overline{-1}|^{n-s+1}$.

* We shall in future leave out the suffixes in ζ_0 and ν_0, using ζ, ν to denote the surface values, as these surface values alone occur in the remainder of our investigations.

In general $F(\kappa)$ does not change sign when $\kappa = -\operatorname{cosec}\alpha$, but when $\kappa = -\infty$, $F(\kappa)$ has the same sign as $\overline{-1}|^{n-s+2}$.

Hence, the equation $F(\kappa) = 0$ must have one real root between each of the following values :—

$$\infty,\ 1,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \ldots\ \kappa_{n-s},\ -\infty.$$

Thus, if $K_n^s(\zeta)$ is positive, all the roots of (66) are real. Let us now examine what happens when $K_n^s(\zeta)$ vanishes and becomes negative. POINCARÉ proves* that, if $t_n^s(\zeta)$ is divisible by ζ, the equation

$$K_n^s(\zeta) = 0$$

has no real root; we must, therefore, have $n - s$ even, so that $t_n^s(\zeta)$ is not divisible by ζ.

When $K_n^s(\zeta)$ vanishes, the equation

$$F(\kappa) = 0$$

reduces to

$$\kappa(\kappa - 1)(1 - \kappa^2 \sin^2\alpha)^{\frac{1}{2}(n-s)} D^s P_n \{\kappa \cos\alpha (1 - \kappa^2 \sin^2\alpha)^{-\frac{1}{2}}\} = 0,$$

of which the roots are

$$0,\ 1,\ \kappa_1,\ \kappa_2,\ \ldots,\ \kappa_{n-s}.$$

Since $n - s$ is even, the equation

$$D^s P_n(\nu) = 0$$

has not zero for one of its roots. Thus, the roots of the period-equation are all real and different. Therefore, when $K_n^s(\zeta)$ changes sign and becomes negative, the period-equation must, at any rate at first, continue to have all its roots real. If it have a pair of complex roots, the ratio of $K_n^s(\zeta) : q_2(\zeta)$ must not only be negative, but numerically greater than some *finite* limit.

16. The roots of the period-equations for the oscillations that are symmetrical about the axis of the spheroid are to be separated in exactly the same way. It will be sufficient to state the results here. We suppose $\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_n$ are the n values of κ which make

$$P_n(\nu) \equiv P_n\{\kappa \cos\alpha (1 - \kappa^2 \sin^2\alpha)^{-\frac{1}{2}}\} = 0 \quad \ldots \quad (71).$$

Let n be odd. One of the above values, viz., $\kappa_{\frac{1}{2}(n+1)}$ will be zero, whilst $\kappa_n = -\kappa_1$, $\kappa_{n-1} = -\kappa_2$, and so on. Also, the ratio $K_n(\zeta) : q_2(\zeta)$ *must* be positive. It will be found that the period-equation (64) has one root between each of the following values of κ;

$$\infty,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \ldots,\ \kappa_{\frac{1}{2}(n-1)},\ 0,\ \kappa_{\frac{1}{2}(n+3)},\ \ldots,\ \kappa_n,\ -\infty.$$

* 'Acta Mathematica,' vol. 7, p. 326.

If n be even, the least positive and negative roots of (71) are $\kappa_{\frac{1}{2}n}$ and $\kappa_{\frac{1}{2}n+1}$, also $\kappa_{\frac{1}{2}n+1} = -\kappa_{\frac{1}{2}n}$. If the ratio $K_n(\zeta) : q_2(\zeta)$ be positive, we find that the positive roots of the period-equation (62) are situated between the following values :—

$$\infty, \kappa_1, \kappa_2, \ldots, \kappa_{\frac{1}{2}n},$$

while the negative ones which are equal and opposite to them are situated in the intervals between

$$-\infty, \kappa_n, \kappa_{n-1}, \ldots, \kappa_{\frac{1}{2}n+1},$$

there being no roots between $\kappa_{\frac{1}{2}n}$ and $\kappa_{\frac{1}{2}n+1}$.

When $K_n(\zeta)$ vanishes, the roots are the n quantities

$$\kappa_1, \kappa_2, \ldots, \kappa_n,$$

none of which is equal to zero. If the ratio $K_n(\zeta) : q_2(\zeta)$ now become negative, the roots of (62) will at first continue to be real, being situated between the values

$$\kappa_1, \kappa_2, \ldots, \kappa_{\frac{1}{2}n}, 0, \kappa_{\frac{1}{2}n+1}, \ldots, \kappa_n.$$

This will be the case until we arrive at a value of ζ for which the period-equation has a pair of equal roots, each equal to zero. When this is so, we have

$$\frac{K_n(\zeta)}{4q_2(\zeta)} = \mathrm{L}^t_{\nu=0} \frac{\nu P_n(\nu)}{DP_n(\nu)} = -\frac{1}{n(n+1)},$$

whence,

$$K_n(\zeta) = -\frac{4}{n(n+1)} q_2(\zeta) = \frac{4}{n(n+1)} K_1^1(\zeta) \quad . \quad . \quad . \quad . \quad (72),$$

or

$$p_1(\zeta)\, q_1(\zeta) - p_n(\zeta)\, q_n(\zeta) = \frac{4}{n(n+1)} \{p_1(\zeta)\, q_1(\zeta) - t_1^1(\zeta)\, u_1^1(\zeta)\}.$$

When the ratio $K_n(\zeta) : -q_2(\zeta)$ or $K_n(\zeta) : K_1^1(\zeta)$ becomes greater than $4/\{n(n+1)\}$, two of the roots of the period-equation will become imaginary.

In *every* case there must be at least one positive root between each of the quantities

$$\kappa_1, \kappa_2, \ldots, \kappa_{\frac{1}{2}n}$$

and corresponding negative roots, so that under no circumstances can equation (62) have *more* than one pair of imaginary roots.

Numerical Solutions of the Period-Equations.

17. For a spheroid of given eccentricity, α, and therefore ζ, are known. Now, the functions $p_n(\zeta)$, $t_n^s(\zeta)$ can be expanded in finite terms of ζ in exactly the same way

as the ordinary spherical harmonics, while $q_n(\zeta)$, $u_n^s(\zeta)$ can be expressed in finite terms of ζ, $\cot^{-1}\zeta$, *i.e.*, of ζ, α; hence, the function $K_n^s(\zeta)$ can be calculated for any value of ζ.* By HORNER'S method we may then approximate to the values of the roots of the equation in κ in the simpler cases. The periods of the waves are the corresponding values of $\pi/\kappa\omega$, while ω is expressed in terms of ρ by equation (50).†

To obtain some idea of the relative frequencies of the various waves, I have tabulated the values of κ thus calculated for harmonics of the second, third, and fourth degrees for a spheroid in which $\zeta = 1$ or $\alpha = \pi/4$, the eccentricity being, therefore, $\frac{1}{2}\sqrt{2}$. The results are embodied in the accompanying Table. As already stated, the positive roots correspond to waves rotating more slowly than the liquid, or relatively in the direction opposite to that of rotation of the mass, while those having the double sign correspond to symmetrical oscillations of the liquid.

TABLES of the Values of $\kappa\left(=\frac{\lambda}{2\omega}\right)$ for Waves on a Spheroid whose Eccentricity $= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Rank of Harmonic.	I. Harmonics of the Second Degree.				
2 (sectorial)	1·2126108,	− 0·2126108.			
1	1·280776,	− 0·780776.			
0 (zonal) Oscillatory waves	± 1·128465.				
	II. Harmonics of the Third Degree.				
3 (sectorial)	1·569830,	− 0·569830.			
2	1·677377,	0·423263,	− 1·100640.		
1	1·6008928,	0·8267846,	− 0·0733654,	− 1·3543120.	
0 (zonal) Oscillatory waves	± 1·5178954,	± 0·5122368.			
	III. Harmonics of the Fourth Degree.				
4 (sectorial)	1·852560,	− 0·852560.			
3	1·806374,	0·366650,	− 1·173024.		
2	1·921878,	0·730910,	− 0·107365,	− 1·545423.	
1	1·924662,	0·890193,	0·585670,	− 0·654623,	− 1·745902.
0 (zonal)	± 1·994751,	± 0·685895.			

I find that the period of the symmetrical or zonal harmonic oscillation of the second degree (in which the surface remains spheroidal) is, in this spheroid, 0·8599258

* "On the Expression of Spherical Harmonics of the Second Kind in a Finite Form," 'Cambridge Philosophical Proceedings,' December, 1888.

† See THOMSON and TAIT'S 'Natural Philosophy,' vol. 2, § 772.

of the corresponding time of oscillation in a non-rotating spherical mass of liquid of the same density.

Sectorial Harmonic Waves.

18. When $s = n$, $D^sP_n(\nu)$ is numerical, and $D^{s+1}P_n(\nu)$ is zero; thus, the period-equation reduces to

$$4\kappa(\kappa - 1) = nK_n^n(\zeta)/q_2(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

of which the roots are given by

$$\kappa = \tfrac{1}{2}\{1 \pm \surd(1 + nK_n^n(\zeta)/q_2(\zeta))\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

The condition that these roots may be real is that

$$q_2(\zeta) + nK_n^n(\zeta) > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75),$$

that is

$$p_1(\zeta)\,q_1(\zeta) - t_n^n(\zeta)\,u_n^n(\zeta) - \frac{1}{n}\{p_1(\zeta)\,q_1(\zeta) - t_1^1(\zeta)\,u_1^1(\zeta)\}$$

must be positive.

These results have been obtained previously by POINCARÉ in the special case in which $n = 2$, but in his investigation an extraneous factor has been introduced into the period-equation, giving a third root ($\kappa = 1$) which does not properly belong to it.

The expression for ψ in (21) is here proportional to

$$(1 - \mu'^2)^{\frac{1}{2}n}(\nu^2 - 1)^{\frac{1}{2}n}\,e^{\iota(n\phi + 2\omega\kappa t)},$$

that is, in Cartesian coordinates, to

$$(x + \iota y)^n\,e^{2\iota\omega\kappa t},$$

and is independent of z.

Thus, the motion of the liquid is "two dimensional," and takes place in planes parallel to the equatorial plane of the spheroid. By the laws of vortex motion the molecular rotation or spin of the actual motion of the liquid is therefore everywhere constant and equal to ω, being that due to the rotation of the liquid. In other words, the wave motion of the liquid *relative* to the rotating axes is irrotational.

Small Free Precession of the Spheroid.

19. Another case of some interest is when the harmonics determining the small periodic relative motions are of the second degree and first rank. Putting $n = 2$, $s = 1$, the period-equation (66) reduces to

$$4\kappa^2(\kappa - 1)\,q_2(\cot\alpha) - [(\kappa - 1)\sec^2\alpha + \kappa]\,K_2^1(\cot\alpha) = 0 \quad . \quad . \quad (76).$$

Now, whatever the value of α may be, $\kappa = \frac{1}{2}$ is always a root of this equation. For, if we put $\kappa = \frac{1}{2}$ in the left-hand side, it becomes

$$\begin{aligned} &= -\tfrac{1}{2} q_2(\cot\alpha) + \tfrac{1}{2}\tan^2\alpha\, \mathrm{K}_2{}^1(\cot\alpha) \\ &= \tfrac{1}{2}\{\mathrm{K}_1{}^1(\zeta) + \mathrm{K}_2{}^1(\zeta)/\zeta^2\} \\ &= \tfrac{1}{2}\{(\zeta^2+1)p_1(\zeta)\,q_1(\zeta)/\zeta^2 - t_1{}^1(\zeta)\,u_1{}^1(\zeta) - t_2{}^1(\zeta)\,u_2{}^1(\zeta)/\zeta^2\} \\ &= \tfrac{1}{2}(\zeta^2+1)\int_\zeta^\infty \left\{\frac{1}{\zeta^2} - \frac{1}{\zeta^2+1} - \frac{1}{\zeta^2(\zeta^2+1)}\right\}\frac{d\zeta}{\zeta^2+1} = 0 \quad . \quad . \quad . \quad . \quad (77), \end{aligned}$$

as was to be proved.

Substituting from the relation just found in (76), the period-equation becomes

$$(4\kappa^3 - 4\kappa^2 + \kappa)\tan^2\alpha - (2\kappa - 1)\sec^2\alpha = 0,$$

or, dividing throughout by $(2\kappa - 1)\tan^2\alpha$, the other two roots are given by

$$2\kappa^2 - \kappa - \operatorname{cosec}^2\alpha = 0,$$

whence

$$\begin{aligned} \kappa &= \tfrac{1}{4}\{1 \pm \sqrt{(1 + 8\operatorname{cosec}^2\alpha)}\} \\ &= \tfrac{1}{4}\{1 \pm \sqrt{(9 + 8\zeta^2)}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78). \end{aligned}$$

The expression for ψ is proportional to

$$(1 - \mu'^2)^{\frac{1}{2}}\mu'(\nu^2 - 1)^{\frac{1}{2}}\nu\sin(\phi + 2\omega\kappa t - \epsilon),$$

that is, to

$$z\{x\sin(2\omega\kappa t - \epsilon) + y\cos(2\omega\kappa t - \epsilon)\},$$

while the height of the displacement of the surface is proportional to

$$\varpi z\{x\sin(2\omega\kappa t - \epsilon) + y\cos(2\omega\kappa t - \epsilon)\}.$$

Remembering that this displacement is so small that its square may be neglected, it can be readily shown by the usual methods of analytical geometry that, if, as is here supposed, the ellipticity of the spheroid be finite, the displaced surface is a spheroid of the same form and dimensions as the original spheroid, and can be obtained by turning the latter through a small angle about the line

$$x\sin(2\omega\kappa t - \epsilon) + y\cos(2\omega\kappa t - \epsilon) = 0, \qquad z = 0.$$

This will, however, no longer be true if the ellipticity of the spheroid is a small quantity comparable with the height of the small displacement, or the surface is spherical or nearly spherical. In such cases it will be found that the displaced surface is an ellipsoid, differing in form from the original spheroid by small quantities of the first order, whose axes make finite, not small, angles with those of the spheroid.

Suppose the liquid spheroid is rotating steadily about its axis of figure with angular velocity ω, and that this axis does not quite coincide with our fixed axis of z, but is inclined to it at a small angle, while the axes of x, y rotate about the axis of z with angular velocity ω. The coordinates of the fluid particles will now no longer be constant, but will undergo small periodic changes. In the time $2\pi/\omega$ both the fluid particles and the axes will come round to their original positions; thus, the period of the *apparent* relative oscillations is $2\pi/\omega$, although the liquid is in reality rotating steadily. This accounts for the root $\kappa = \frac{1}{2}$, which occurs in the period-equation, a result which may be completely verified by rigorous analytical methods.

The movements corresponding to the other two roots are somewhat similar to precession, the axis of figure of the spheroid turning about the axis of z, to which it is inclined at a small angle.

Stability of the Spheroid.

20. We have already alluded to POINCARÉ'S investigations of the condition that the spheroid, if viscous, may be secularly stable, which requires that the energy of the system for the given angular momentum must be a minimum in the spheroidal form. The greatest eccentricity corresponds to the least value of ζ which causes any one of the coefficients $K_n^s(\zeta)$ to vanish, and this is shown to be that given by $K_2^2(\zeta) = 0$, whence, as in THOMSON and TAIT (§ 772),

$$1/\zeta = \tan\alpha = f = 1{\cdot}39457.$$

If the liquid be perfectly inviscid, the criteria are very different. So long as the roots of the period-equations for the various waves and oscillations are all real, the spheroid cannot be unstable. It will, however, become unstable if for any harmonic the equation in κ has a pair of complex or imaginary roots. For, calling these roots $l \pm m\iota$, we get the possible surface displacements

$$h = C_n^s\, \varpi\, T_n^{(s)}(\mu)\, e^{\iota s\phi} e^{2\omega(\iota l + m)t},$$
$$h = C_n^s\, \varpi\, T_n^{(s)}(\mu)\, e^{-\iota s\phi} e^{2\omega(-\iota l + m)t},$$

compounding into the displacement

$$h = C_n^s\, \varpi\, T_n^{(s)}(\mu) \cos(s\phi + 2\omega l t)\, e^{2\omega m t},$$

which increases indefinitely with the time.

Let us imagine that our spheroid is subject to constraints such as to freely allow of its surface undergoing harmonic displacements of degree n and rank s, but which allow of no other displacements (such constraints are, of course, purely theoretical). The spheroid, if at all viscous, will be secularly stable or unstable according as

$$K_n^s(\zeta) > 0 \quad \text{or} \quad < 0,$$

and we have seen that the latter condition can only hold if $n - s$ be even, that is, if the displacement be one symmetrical with respect to the equatorial plane. The critical form is that in which

$$K_n^s(\zeta) = 0.$$

But since, when $K_n^s(\zeta)$ changes sign, the roots of the period-equation at first continue real, the limits of eccentricity for which the perfect spheroid is "ordinarily" stable are in every case wider than those consistent with secular stability if the liquid be viscid. The critical form is determined by the condition that the period-equation must have a pair of equal roots.

In my paper on "The Waves on a Viscous Rotating Cylinder"* I have endeavoured to further elucidate the difference between "ordinary" and "secular" stability. Assuming the displacement from relative equilibrium to be proportional to $e^{-\alpha_1 t}$, α_1 is always complex for viscous liquid, and the condition that the disturbance may not increase with the time is that the real part of α_1 must be positive. Both real and imaginary parts change sign when $\alpha_1 = 0$, the corrugations becoming relatively fixed and the liquid figure becoming a form of "bifurcation." Relative equilibrium is then critical. But, if there be no viscosity, α_1 may be purely imaginary, as in the present case, when κ is real, and the waves will neither increase nor diminish in amplitude with the time; thus, a change in the sign of one of the roots of the period-equation merely implies a change in the relative direction of the wave.

Moreover, it appears that the criteria of ordinary and secular stability will be different only if the angular velocities of the waves be different in the two opposite directions, and this can only be the case if the liquid be rotating.

Reverting to the perfect liquid spheroid, the determination of the greatest eccentricity consistent with ordinary stability involves the question, if ζ be gradually diminished, what is the harmonic displacement for which the period-equation of the waves first commences to have complex roots? It appears probable that this happens for $n = 2$, $s = 2$. With this assumption, we see, by (75), that the critical value of ζ is given by the equation

$$2\{p_1(\zeta)\,q_1(\zeta) - t_2^2(\zeta)\,u_2^2(\zeta)\} + t_1^1(\zeta)\,u_1^1(\zeta) - p_1(\zeta)\,q_1(\zeta) = 0\,;$$

this leads to

$$(3\zeta^4 + 8\zeta^2 + 1)\cot^{-1}\zeta - (3\zeta^3 + 7\zeta) = 0,$$

* 'Cambridge Philosophical Proceedings,' 1888.

whence I find by trial and error that

$$1/\zeta = \tan\alpha = f = 3{\cdot}1414567 \ldots .$$

This result agrees with that found by RIEMANN, who treated the problem as a special case of the general motion of a liquid ellipsoid.

It does not, however, seem possible to justify the above assumption as to the nature of the displacements by a perfectly general rigorous proof. The condition that the period-equation should have a pair of equal roots is far too complicated to allow of this point being fully proved in the way that POINCARÉ has done for secular stability. It is certain that the spheroid will be unstable for all values of $\tan\alpha$ greater than $3{\cdot}1414567$; it is probable, but not certain, that it will be stable for all values less than this limit.

Spheroids of Small Ellipticity.

21. If the eccentricity of the spheroid, and, therefore also, its angular velocity, be small, the period-equations for the waves are much modified. The value of ζ will become very great, and we shall suppose it to be so great that ζ^{-2} is a small quantity that can be neglected. Since α is small, we may put $\cos\alpha = 1$, $\sin\alpha = \alpha = 1/\zeta$.

The function $t_n^s(\zeta)$ is proportional to ζ^n since the other terms in it involve only ζ^{n-2} and lower powers of ζ. Hence, to this approximation,

$$t_n^s(\zeta)\, u_n^s(\zeta) = \zeta^{2n}\int_\zeta^\infty \frac{d\zeta}{\zeta^{2n+2}} = \frac{\zeta^{-1}}{2n+1} \quad \ldots \ldots \quad (79),$$

and

$$K_n^s(\zeta) = \zeta^{-1}\left(\tfrac{1}{3} - \frac{1}{2n+1}\right) = \frac{2(n-1)}{3(2n+1)}\zeta^{-1} \quad \ldots \ldots \quad (80);$$

moreover,

$$q_2(\zeta) = -K_1^1(\zeta) = \tfrac{2}{15}\zeta^{-3} \quad \ldots \ldots \quad (81),$$

whence, as in THOMSON and TAIT (§ 771),

$$\omega^2 = \tfrac{8}{15}\pi\rho\gamma\,\zeta^{-2} \quad \ldots \ldots \quad (82).$$

Firstly, suppose that the values of κ remain finite in the limit. Then $\nu = \kappa$ ultimately, and, since $q_2(\zeta)$ is negligible in comparison with $K_n^s(\zeta)$, equation (65) gives

$$s D^s P_n(\kappa) + (\kappa - 1) D^{s+1} P_n(\kappa) = 0 \quad \ldots \ldots \quad (83),$$

having $n - s$ real roots between 1 and -1.

In the case of the oscillations symmetrical about the axis ($s = 0$) the equation for κ is ultimately

$$DP_n(\kappa) = 0 \quad \text{or} \quad \frac{1}{\kappa} DP_n(\kappa) = 0 \quad \ldots \ldots \quad (84),$$

according as n is odd or even.

The frequencies of these waves or oscillations are proportional to the angular velocity of the liquid. As the latter is diminished without limit they become relatively unimportant, and finally cease to exist, for the limiting case of a mass of liquid without rotation oscillating about the spherical form.

Secondly, suppose the periods of the waves remain finite in the limit. Put $2\omega\kappa = \lambda$, so that $2\pi/\lambda$ is the period. Since λ remains finite, κ will increase without limit as ω diminishes, and, therefore, equation (66) gives, to the first order of small quantities,

$$\frac{2\zeta^{-2}}{15}\frac{\lambda(\lambda - 2\omega)}{\omega^2} - \frac{2}{3}\frac{(n-1)}{(2n+1)}s - \frac{2}{3}\frac{(n-1)}{(2n+1)}(n-s)\frac{\lambda - 2\omega}{\lambda} = 0,$$

whence, by (82),

$$\lambda(\lambda - 2\omega) = \frac{8n(n-1)}{3(2n+1)}\pi\rho\gamma\left\{1 - \frac{n-s}{n}\cdot\frac{2\omega}{\lambda}\right\} \quad \ldots \quad (85).$$

If we put $\omega = 0$, we get

$$\lambda^2 = \frac{8n(n-1)}{3(2n+1)}\pi\rho\gamma \quad \ldots \quad (86),$$

the well-known result for the oscillations of a liquid sphere. Denoting by Λ^2 the expression

$$\frac{8}{3}\frac{n(n-1)}{(2n+1)}\pi\rho\gamma,$$

we find

$$\lambda^2 - \Lambda^2 = \frac{2\omega}{\lambda}\left(\lambda^2 - \frac{n-s}{n}\right),$$

whence, substituting $\lambda = \pm\Lambda$ in the small terms, we get

$$\lambda = \pm\Lambda + \frac{s}{n}\omega \quad \ldots \quad (87).$$

Remembering the expressions found in § 13 for the relative and actual angular velocities of the corresponding waves, this result may be stated as follows :—The effect of communicating a small angular velocity ω to a spherical mass of gravitating liquid will be to add an angular velocity $(n-1)\omega/n$ to the angular velocities of all the free waves which are determined by harmonics of degree n.

The symmetrical oscillations will be unaffected by rotation to this order of approximation. If we proceed to a higher approximation by taking into account small quantities of the second order, the equations become much more complicated. But, for a spheroid similar to the Earth, the above approximation would be practically sufficient.

Forced Tides.

22. We now revert to the applications of the methods of this paper to the investigation of the tides produced on the surface of the spheroid by the influence of periodic variations of pressure over the surface of the liquid, or by the attractions of disturbing bodies in the neighbourhood of the spheroid.

In this connexion, the equations found in §§ 9, 10 will be required, viz., if at the surface

$$V_2 - p_2/\rho = \Sigma\Sigma\Sigma W^s_{(n,\kappa)} T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n\kappa}{}^s) \quad . \quad . \quad . \quad . \quad (88),$$

and

$$h = \varpi\, \Sigma\Sigma\Sigma C^s_{(n,\kappa)} T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n\kappa}{}^s) \quad . \quad . \quad . \quad . \quad (89),$$

then $C^s_{(n,\kappa)}$ will be given in terms of $W^s_{(n,\kappa)}$ by the relation

$$3\frac{M\gamma}{c} C^s_{(n,\kappa)} \left\{ K_n{}^s(\zeta) - \frac{4\kappa D^s P_n(\nu)}{sD^s P_n(\nu)/(\kappa-1) + \sec^2\alpha \,.\, \nu D^{s+1} P_n(\nu)/\kappa} q_2(\zeta) \right\} = W^s_{(n,\kappa)} \quad (90),$$

where, as in (12), (15),

$$\left. \begin{aligned} \nu &= \frac{\kappa \cos\alpha}{\sqrt{(1-\kappa^2 \sin^2\alpha)}}, \\ \text{and} \qquad \zeta &= \cot\alpha \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91).$$

Also, from (39), we have

$$12\frac{M\gamma}{c} C^s_{(n,\kappa)} \frac{4\kappa D^s P_n(\nu)\, q_2(\zeta)}{sD^s P_n(\nu)/(\kappa-1) + \sec^2\alpha \,.\, \nu D^{s+1} P_n(\nu)/\kappa} = -A^s_{(n,\kappa)} T_n{}^s(\nu) \quad . \quad (92).$$

The value of ψ at any point of the surface being

$$[\psi] = A^s_{(n,\kappa)} T_n{}^s(\nu)\, T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n,\kappa}{}^s) \quad . \quad . \quad . \quad . \quad (93),$$

is determined in terms of $C^s_{(n,\kappa)}$ by the last equation (92).

23. An interesting case occurs when $\kappa = 1$. The period of the tides will then be half that of a complete revolution of the liquid, and they may therefore be called "semidiurnal" with reference to the spheroid. Except in the case when $s = 0$, equation (90) gives

$$3\frac{M\gamma}{c} C_{(n)}{}^s K_n{}^s(\zeta) = W_n{}^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94);$$

also, from (92), $A_n{}^s T_n{}^s(\nu) = 0$, and, therefore, $[\psi] = 0$; hence, it is evident that ψ must also vanish throughout the liquid.

The height of the forced tides is therefore the same as we should get by the "equilibrium theory," *i.e.*, by neglecting the small relative motions of the fluid particles entirely. In fact, these relative motions have no effect on the height of the tides. It does not follow that they do not exist; in fact, it is evident, on the contrary, since the tides move relatively to the liquid, that they must exist. But on referring to equations (34) we see that, when $\kappa = 1$, the small relative velocity components U, V, W may be finite, even though ψ vanishes.

The zonal oscillations are, however, given by

$$3\frac{M\gamma}{c}C_n\left\{K_n(\zeta) - \frac{8\cos^2\alpha\, q_2(\zeta)}{n(n+1)}\right\} = W_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

since $\nu = 1$, and therefore $P_n(\nu) = 1$, $DP_n(\nu) = \frac{1}{2}n(n+1)$. For these oscillations ψ does not vanish.

24. Another interesting application is to determine the height of the permanent corrugations produced by disturbing forces which remain constant and fixed relatively to the rotating liquid. We now have to take $\kappa = 0$; therefore, $\nu = 0$ and $\nu/\kappa = \cos\alpha$.

If s is different from zero, then, whether $n - s$ be odd or even, equation (90) gives us

$$3(M\gamma/c)C_n^sK_n^s(\zeta) = W_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94),$$

and (92) gives $A_n^sT_n^s(\nu) = 0$, whence $[\psi] = 0$; and therefore ψ is everywhere zero.

If $s = 0$ and n is odd, then $P_n(0) = 0$, $DP_n(0)$ is finite, and, as before, we find

$$3(M\gamma/c)C_nK_n(\zeta) = W_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94\text{A}),$$

and

$$\psi = 0.$$

Lastly, let $s = 0$ and let n be even. This is the case of a harmonic disturbance which is symmetrical both about the axis of the spheroid and also with respect to its equatorial plane. Then, as in § 16, $4\nu P_n(\nu)/DP_n(\nu)$ approaches the finite limit $-4/\{n(n+1)\}$, when ν is diminished indefinitely, and therefore

$$3\frac{M\gamma}{c}C_n\left\{K_n(\zeta) - \frac{4q_2(\zeta)}{n(n+1)}\right\} = W_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96),$$

$$[\psi] = -\frac{12}{n(n+1)}\frac{M\gamma}{c}C_nq_2(\zeta)P_n(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

In the first two cases the height of the corrugations is given by the "equilibrium theory," and, since $\psi = 0$, it follows from equations (34) that U, V, W are all zero. Thus, the fluid continues to rotate as if rigid in a form differing slightly from the original spheroid, as we should most naturally expect. But in the last case, since ψ is finite, there will be a finite relative motion of the liquid with respect to the moving axes. That this must be the case may be seen as follows :—The spheroid is supposed to be deformed from its original form by the action of the given conservative forces and surface pressures. The displacement does not vanish at the equator; hence, if we consider the fluid particles at the surface, forming a circle round the equator, the displacement must necessarily increase or diminish the size of this circle. By THOMSON'S circulation theorem, the circulation in this circuit must remain the same as before, since the liquid is supposed perfect; hence, the angular velocities of the fluid particles in this circle must be altered, and they can no longer continue to rotate about the axis of the spheroid with the original angular velocity ω. Therefore the disturbance must produce permanent relative motions of the liquid, unless there be any viscosity present, in which case the mass will ultimately rotate as if rigid in the deformed figure, and the "equilibrium theory" will again become applicable.

Tides due to Action of a Satellite.

25. We shall conclude by showing how to determine the forced tides due to the presence of a small satellite of mass m revolving in any orbit about the spheroid.

If we take ϕ' to be the longitude of any point on the spheroid, measured from a plane fixed in space, with which the moving plane of (y, z) coincides at time $t=0$, then, ϕ being the longitude measured from the latter plane, we have

$$\phi' = \phi + \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98).$$

Let $(\mu_1, \zeta_1, \phi'_1)$ be the spheroidal coordinates of the mass m at time t, ϕ'_1 being measured from the fixed initial plane. Then, at any point (μ, ζ, ϕ') whose distance from the mass is R, we have

$$V_2 = m\gamma/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99).$$

Since $\zeta_1 > \zeta$, $1/R$ can be expanded in spheroidal harmonics by the formula

$$1/R = 1/c \sum_{n=0}^{n=\infty} (2n+1) [P_n(\mu) p_n(\zeta) P_n(\mu_1) q_n(\zeta_1)$$
$$+ 2 \sum_{s=1}^{s=n} \{(n-s)!/(n+s)!\} T_n^{(s)}(\mu) t_n^s(\zeta) T_n^{(s)}(\mu_1) u_n^s(\zeta_1) \cos s(\phi' - \phi'_1)] \quad (100).$$

Since the motion of the satellite is supposed known, $(\mu_1, \zeta_1, \phi'_1)$ are known functions of t. In order to complete the solution we must suppose the quantities $P_n(\mu_1) q_n(\zeta_1)$, $T_n^{(s)}(\mu_1) u_n^s(\zeta_1) \cos s\phi'_1$, and $T_n^{(s)}(\mu_1) u_n^s(\zeta_1) \sin s\phi'_1$ expanded by FOURIER'S theorem in simple harmonic functions of the time. If the period of the satellite in its orbit be $2\pi/L$, the expansion will only involve circular functions of multiples of Lt. We then write $\phi + \omega t$ for ϕ', and resolve all *products* of sines and cosines involving ϕ or t in terms of circular functions of sums and differences. The expression V_2 will now be a triple series of the same form as in equation (88), and the heights of the tides due to each term of the series may be determined separately by the application of equations (89), (90).

26. Suppose that the attracting mass rotates round the spheroid in the equatorial plane in a circle of radius $c\sqrt{(Z^2+1)}$ with angular velocity $(1-2l)\,\omega$ (referred to fixed axes).

We then have $\mu_1 = 0$, $\zeta_1 = Z$, $\phi'_1 = (1-2l)\,\omega t$, $\phi' - \phi'_1 = \phi + 2l\omega t$. Therefore

$$V_2 = m\gamma/c\,\Sigma_{n=1}^{n=\infty}(2n+1)\,[P_n(\mu)\,p_n(\zeta)\,P_n(0)\,q_n(Z) + 2\Sigma_{s=1}^{s=n}\{(n-s)!/(n+s)!\}\,T_n^{(s)}(\mu)\,t_n^s(\zeta)\,T_n^{(s)}(0)\,u_n^s(Z)\cos s\,(\phi + 2l\omega t)] \quad (101).$$

We have

$$P_n(0) = 0, \ (n \text{ odd}),$$

$$P_n(0) = \frac{(-1)^{\frac{1}{2}n}.\,n!}{2^n\,(\frac{1}{2}n!)^2}, \ (n \text{ even}),$$

$$T_n^{(s)}(0) = 0, \ (n-s \text{ odd}),$$

$$T_n^{(s)}(0) = \frac{(-1)^{\frac{1}{2}(n+s)}\,(n+s)!}{2^n.\frac{1}{2}(n+s)!.\frac{1}{2}(n-s)!}, \ (n-s \text{ even}),$$

so that the expansion only involves harmonics which are symmetrical with respect to the equatorial plane.

The first term in the above expansion is a harmonic of the first degree and rank, and determines the motion of the spheroid as a whole about the centre of mass of the spheroid and the attracting body. This motion can be taken separately.

Taking, in the usual way,

$$h = \varpi\,\Sigma_{n=2}^{n=\infty}\{C_nP_n(\mu) + \Sigma_{s=1}^{s=n}C_n^sT_n^{(s)}(\mu)\cos s\,(\phi + 2l\omega t)\} \quad . \quad . \quad . \quad (102),$$

where the summation includes only even values of $n-s$, we find, by the method of § 24,

$$C_n = \frac{2n+1}{3}\,\frac{m}{M}\,\frac{p_n(\zeta)\,P_n(0)\,q_n(Z)}{K_n(\zeta) - 4q_2(\zeta)/\{n(n+1)\}}$$

$$= (-1)^{\frac{1}{2}n}.\tfrac{1}{3}(2n+1)\frac{m}{M}\,\frac{n!}{2^n\,(\frac{1}{2}n!)^2}\cdot\frac{p_n(\zeta)\,q_n(Z)}{K_n(\zeta) - 4q_2(\zeta)/\{n(n+1)\}}, \ (n \text{ even}) \quad . \quad (103);$$

also, if $n-s$ be even,

$$C_n^s = (-1)^{\frac{1}{2}(n+s)}\tfrac{2}{3}(2n+1)\frac{m}{M}\cdot\frac{n-s!}{2^n.\frac{1}{2}(n+s)!\,\frac{1}{2}(n-s)!}\,t_n^s(\zeta)\,u_n^sZ$$

$$\div\left\{K_n^s(\zeta) - \frac{4ls\,(ls-1)\,q_2(\zeta)\,D^sP_n(\nu_s)}{sD^sP_n(\nu_s) - (ls-1)\sec^2\alpha\,.\,\nu_sD^{s+1}P_n(\nu_s)/(ls)}\right\} \quad (104),$$

where, in this case,

$$\nu_s = ls \cos \alpha \, (1 - l^2 s^2 \sin^2 \epsilon)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105),$$

the corresponding value of κ being ls.

Equations (102), (103), (104) fully determine the height of the forced tides and deformations on the surface of the spheroid due to the attracting body.

If, instead of one attracting mass m, we suppose two equal masses $\frac{1}{2}m$ on opposite sides of the body, the expression for V_2, and therefore for h, will contain only zonal harmonics, and harmonics of even rank, but these will be the same as before.

Harmonic Tides of the Second Order.

27. If the body be very distant from the spheroid, Z will be great, and the series (101) will converge very rapidly, so that the harmonics of the second degree are the most important.

Taking $n = 2$, $s = 2$, we get

$$C_2^2 = \frac{5}{12} \frac{m}{M} \cdot \frac{t_2^2 (\zeta) \, u_2^2 (Z)}{K_2^2 (\zeta) - 4l (2l - 1) \, q_2 (\zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (106),$$

and the corresponding term in the height of the forced tide is

$$h_2^2 = \frac{15}{4} \frac{m}{M} \frac{(\zeta^2 + 1) \, u_2^2 (Z)}{K_2^2 (\zeta) - 4l (2l - 1) \, q_2 (\zeta)} (1 - \mu^2) \, \varpi \cos 2 \, (\phi + 2m \, \omega t) \quad . \quad (107).$$

If the attracting body be rotating in the same direction as the liquid, but with less angular velocity (as in all cases of astronomical interest), $2l$ lies between 0 and 1. If, in addition, $K_2^2 (\zeta)$ is positive, the denominator in the expression for h_2^2 cannot vanish, and the tide produced by this term cannot therefore become very large. In fact, the angular velocities of both the free harmonic tides will lie beyond the above-mentioned limits, and will neither of them coincide with that of the attracting body. If, however, $K_2^2 (\zeta) = 0$, and if, in addition, the attracting body be fixed in space, so that $2l = 1$, this forced tide will increase indefinitely. The spheroid will now have a semi-diurnal free tide, which will be fixed in space, and will coincide with the forced tide due to the attracting body. The equilibrium in the spheroidal form will therefore be completely broken up.

If the attracting body be rotating very slowly about the spheroid, the same thing will happen if $K_2^2 (\zeta)$ has a certain corresponding small negative value.

Since the spheroid is secularly stable or unstable according as $K_2^2 (\zeta)$ is positive or negative, the mode in which its relative equilibrium will be destroyed if $K_2^2 (\zeta)$ becomes negative will depend on circumstances. If the liquid possess but little viscosity, the changes due to secular instability will take place very slowly, but the effect of the tide generating force due to an attracting body when the angular velocity of one of the free tides coincides with that of the body will become very great. If, however, the viscosity be considerable, it will prevent the forced tides from becoming large, and will cause the liquid to rapidly assume the form of a JACOBIAN ellipsoid, owing to the spheroidal form being secularly unstable.

Conclusion.

28. The results of the present paper suggest several considerations, which may possibly throw further light on the past history of the Solar system. The criteria of stability applicable to the two cases where the spheroid is formed of perfect and of viscous liquid respectively have been already discussed in § 20. In the last paragraph I have alluded to the possibility that equilibrium in the spheroidal form may be broken up by an attracting body which causes the harmonic tides of the second order to increase indefinitely, in accordance with Professor DARWIN'S hypothesis.

The present analysis, however, suggests that the same thing may happen in the case of harmonic tides of higher order than the second, and, moreover, the results arrived at concerning the number and situation of the roots of the frequency-equation render this hypothesis quite admissible. Except for harmonics of the second order, many of the free tides will rotate in the same direction as the liquid, but with less angular velocity, even though the spheroid be secularly stable; and, if an attracting body should be rotating about the spheroid with the same angular velocity as one of these tides, they would certainly rise to an enormous height, and the liquid might, perhaps, ultimately be broken up into two or more detached masses.

Take, for example, the sectorial harmonic waves of order n. If one of these be fixed in space, we must, from § 13, have the corresponding value of $\kappa = \frac{1}{2} n$, and, by (73), this leads to the condition

$$K_n^n(\zeta) = (n - 2)\, q_2(\zeta).$$

If n is greater than 2, this will be satisfied for some secularly stable form of the spheroid. Under these circumstances, the presence of a fixed attracting body near the spheroid would cause the sectorial harmonic waves of the n^{th} order to increase indefinitely. The only obstacle in the way of the present supposition is that when the distance of the attracting body is at all considerable, the harmonic components of tide generating force of the higher orders become very small in comparison.

Another question of astronomical interest is whether a rotating spheroid can be broken up into one or more *rings* of rotating liquid. This can only happen if one of the zonal harmonic oscillations increase indefinitely or become unstable.

Now, as long as the spheroid continues stable for such zonal harmonic displacements when the liquid is supposed perfect, the frequencies of these oscillations will none of them vanish ; hence, their amplitude cannot be increased indefinitely by the attractions of bodies remaining in the equatorial plane of the spheroid ; and the only way in which this can take place is under the influence of the tide generating force due to a satellite whose orbit is inclined at a considerable angle to the equatorial plane of the spheroid, the effect being greatest if their planes be perpendicular. There is still the possibility that, contrary to our hypothesis in § 20, a perfect liquid spheroid may first become unstable for some displacement symmetrical about the axis ; and, unless this question be fully decided, we are not in a position to say that such is not the case, and that LAPLACE'S hypothesis is wholly unfounded.

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